Cournot 2 firms

Consider a Cournot oligopoly model in which two firms $i = 1, 2$ simultaneously choose their production levels $q_i \geq 0$. Suppose that firm i's payoff, given the other firm's quantity q_{-i}, is given by

$$\pi_i(q_1, q_2) = q_i \left[10 - (q_i + q_{-i})\right]$$

1. Find the best-response functions, graph them, and determine the Nash equilibrium of the game. Compute total production and the profit obtained by each firm.

2. Find the production level Q^* that maximizes the firms' total profit, obtained by adding both firms' profits, in the Cournot game. Compare each firm's profit under collusion, assuming that each firm produces $q_i = \frac{1}{2}Q^*$ and total profit is split equally, $\pi_i = \frac{1}{2}\Pi^T$, with the profit in the Nash equilibrium.

3. Why is the situation in part 2 not a Nash equilibrium? How is this result consistent with the first welfare theorem?

Solution

1. The profit of firm 1 is

$$\pi_1(q_1, q_2) = q_1[10 - (q_1 + q_2)] = 10q_1 - q_1^2 - q_1q_2$$

The first-order condition is

$$\frac{\partial \pi_1}{\partial q_1} = 10 - 2q_1 - q_2 = 0 \quad \Longrightarrow \quad q_1 = \frac{10 - q_2}{2}$$

Since $q_1 \geq 0$, the best-response function of firm 1 is

$$BR_1(q_2) = \text{máx}\left\{0, \frac{10 - q_2}{2}\right\}$$

By symmetry, the best-response function of firm 2 is

$$BR_2(q_1) = \text{máx}\left\{0, \frac{10 - q_1}{2}\right\}$$

To find the Nash equilibrium, solve

$$q_1 = \frac{10 - q_2}{2} \qquad q_2 = \frac{10 - q_1}{2}$$

Equivalently,

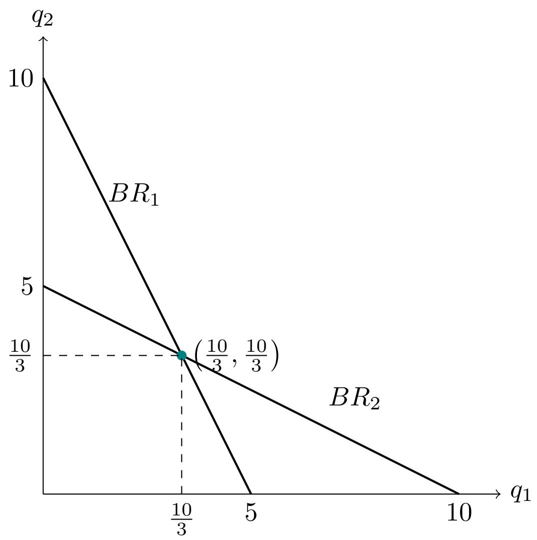
$$2q_1 + q_2 = 10 \qquad q_1 + 2q_2 = 10$$

Subtracting both equations gives $q_1 = q_2$. Hence,

$$3q_1 = 10 \quad \Longrightarrow \quad q_1 = \frac{10}{3} \qquad q_2 = \frac{10}{3}$$

Nash equilibrium: $(q_1^N, q_2^N) = \left(\frac{10}{3}, \frac{10}{3}\right) \approx (3{,}33, 3{,}33)$ The best responses can be graphed as

$$BR_1(q_2) : q_2 = 10 - 2q_1 \qquad BR_2(q_1) : q_2 = \frac{10 - q_1}{2}$$



Total production in the Nash equilibrium is

$$Q^N = q_1^N + q_2^N = \frac{10}{3} + \frac{10}{3} = \frac{20}{3} \approx 6{,}67$$

Each firm's profit is

$$\pi_i^N = \frac{10}{3}\left[10 - \left(\frac{10}{3} + \frac{10}{3}\right)\right] = \frac{10}{3} \cdot \frac{10}{3} = \frac{100}{9} \approx 11{,}11$$

Final results: $Q^N = \frac{20}{3} \approx 6{,}67$ and $\pi_i^N = \frac{100}{9} \approx 11{,}11$ for each firm

2. Total profit is

$$\Pi^T = \pi_1 + \pi_2 = q_1[10 - (q_1 + q_2)] + q_2[10 - (q_1 + q_2)]$$

Taking common factor and defining $Q = q_1 + q_2$,

$$\Pi^T = (q_1 + q_2)[10 - (q_1 + q_2)] = Q(10 - Q) = 10Q - Q^2$$

Maximizing with respect to Q,

$$\frac{d\Pi^T}{dQ} = 10 - 2Q = 0 \quad \Longrightarrow \quad Q^* = 5$$

Since

$$\frac{d^2\Pi^T}{dQ^2} = -2 < 0$$

$Q^* = 5$ maximizes total profit. The maximum total profit is

$$\Pi^T(5) = 10(5) - 5^2 = 25$$

Under collusion, each firm produces half of total output:

$$q_i = \frac{1}{2}Q^* = \frac{5}{2} = 2{,}5$$

Each firm's collusive profit is

$$\pi_i^C = \frac{1}{2}\Pi^T = \frac{25}{2} = 12{,}5$$

Comparing profits,

$$\pi_i^C = \frac{25}{2} = 12{,}5 > \frac{100}{9} \approx 11{,}11 = \pi_i^N$$

Conclusion: under collusion, firms produce less in total and each firm obtains a higher profit than in the Nash equilibrium

3. The collusive outcome is

$$q_1 = q_2 = \frac{5}{2} \qquad \pi_i^C = \frac{25}{2} = 12{,}5$$

To check whether this is a Nash equilibrium, consider firm 1's best response when firm 2 produces $\frac{5}{2}$:

$$BR_1\left(\frac{5}{2}\right) = \frac{10 - \frac{5}{2}}{2} = \frac{15}{4}$$

Thus, if firm 2 produces the collusive quantity, firm 1 prefers to produce $\frac{15}{4}$, not $\frac{5}{2}$. Its profit from deviating is

$$\pi_1\left(\frac{15}{4}, \frac{5}{2}\right) = \frac{15}{4}\left[10 - \left(\frac{15}{4} + \frac{5}{2}\right)\right] = \frac{15}{4} \cdot \frac{15}{4} = \frac{225}{16} \approx 14{,}06$$

Since

$$\frac{225}{16} \approx 14{,}06 > 12{,}5 = \frac{25}{2}$$

firm 1 gains by deviating from the collusive agreement. **Conclusion:** the collusive outcome maximizes joint profit, but it is not individually stable, so $\left(\frac{5}{2}, \frac{5}{2}\right)$ is not a Nash equilibrium This does not contradict the first welfare theorem. The theorem applies to competitive equilibria under specific assumptions, including price-taking behavior. In this Cournot model, firms are not price takers: each firm knows that its own quantity affects the market price

$$p(Q) = 10 - Q$$

Hence, firms have market power and make strategic decisions. Therefore, the Nash equilibrium of the Cournot game need not coincide with the outcome that maximizes the firms' joint profit. **Final conclusion:** there is no contradiction with the first welfare theorem, because the theorem applies to competitive equilibria, not to oligopolies with market power